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Study of Farey Sequences

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Introduction: A Rational Number is one that is expressible as the quotient of two integers. Real numbers that are not rational are called irrational numbers. Farey Fractions give a useful classification of the rational numbers. The Farey Sequence (of counting fractions) has been of interest to modern Mathematicians since the 18th century. Farey Sequence has various properties and applications like clock-making and numerical approximations.

The Farey Sequence, sometimes called the Farey series, is a series of sequences in which each sequence consists of rational numbers ranging in value from 0 to 1. The first sequence, denoted F1 is simply { 0/ 1 , 1/ 1 }. Then to create the nth row we look at the (n − 1)st row and between consecutive fractions $\frac{a}{a'}$ and $\frac{b}{b'}$ insert

$$\frac{a+b}{a'+b'}$$

However the denominator of each term in Fn can be no larger than n. The first five sequences are:

F1= $\{\frac{0}{1},\frac{1}{1}\}$

F2= $\{\frac{0}{1},\frac{1}{2},\frac{1}{1}\}$

F3= $\{\frac{0}{1},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{1}{1}\}$

F4 = $\{\frac{0}{1},\frac{1}{4},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{1}{1}\}$

F5 = $\{\frac{0}{1},\frac{1}{5},\frac{1}{4},\frac{1}{3},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{2}{3},\frac{3}{4},\frac{4}{5},\frac{1}{1}\}$

Note that the terms of each sequence are always increasing in size and are in simplest terms (gcd(a, b) = 1). The Farey sequence appears in many different mathematical entities such as lattices and Ford circles. They can also be used to rationally approximate irrational numbers

History of Farey Sequences: Farey Sequences are named after the British Geologist John Farey, Sr., whose letter about these sequences was published in the Philosophical Magazine in 1816. The story of the Farey sequence and how it came to be is actually quite comical. It is named after John Farey, a geologist from England who was the "first" person to note the properties of rational numbers which make up the sequences. In October of 1801, Farey was out of a job so he returned to London where he published around sixty articles between the years 1804 and 1824 in the magazines Rees's Encyclopaedia, The Monthly Magazine, and Philosophical Magazine. Farey conjectured, without offering proof, that each new term in a Farey Sequence expansion is the mediant of its neighbour. One of the only relevant articles he published was in 1816, titled on a curious property of vulgar fractions. The article consisted of four paragraphs. The first notes the curious property. The second he defines and states the Farey sequence. In the third he gives an example of F5. One of the readers of Farey's article, Augustin-Louis Cauchy, provided the proof in one his writings the same year

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Farey's article was released, and since it was believed that Farey was the first to notice this property the sequence was named after him. Farey in fact was not the first person to observe the properties of the Farey sequence. Charles Haros, in 1802 noticed the property and explained how to construct the 99th sequence. For these reasons Farey is not looked fondly upon in the mathematical community.

I am not acquainted, whether this curious property of vulgar fractions has been before pointed out? ; or whether it may admit of any easy or general demonstration? ; which are points on which I should be glad to learn the sentiments of some of your mathematical readers; and am

Sir,

your obedient humble servant,

J.Farey.

Figure 1.1: Excerpt from Farey's Letter

Farey Sequence And Farey Table: A Farey Sequence Fn is the set of rational numbers p/q with p and q coprime, with 0<p<q<n, ordered by size.

Example: F1= $\{\frac{0}{1}, \frac{1}{1}\}$

F2= $\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\}$

F3= $\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\}$

The Farey sequence of order n is a sequence of irreducible or simple fractions in [0, 1], which have denominators less than then again equivalent to n, organized in climbing request. Every grouping starts with the value 0 (0/ 1) and ends with the value 1 (1 /1). Interestingly, each sequence Fn can be generated from its preceding sequence Fn-1 by inserting the fraction (a + a') esteem (1/1). Strikingly, every arrangement Fn can be produced pair of consecutive fractions a/ b and a'/ b' of Fn-1, disposing of the divisions whose denominators surpass n.

We may note that each sequence has an odd number of terms and the middle term is always ½.

Properties of Farey Sequence: If we have two fractions $\frac{a}{b}$ and $\frac{c}{d}$ with the properties that $\frac{a}{b} < \frac{c}{d}$ and bc-ad=1, then the fractions are known as Farey Neighbours. They appear next to each other in some Farey Sequence. The mediant or Freshman's sum of these two fractions is given by

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$$

and this mediant is the unique fraction with smallest denominator.

Mediant Propererty: If $\frac{a}{b} < \frac{c}{d}$ then their mediant $\frac{a+c}{b+d}$ lies between them because $\frac{a+c}{b+d} - \frac{a}{b} = \frac{bc-ad}{b(b+d)} > 0$ and $\frac{c}{d} - \frac{a+c}{b+d} = \frac{bc-ad}{d(b+d)} > 0$.

Consequently, if two fractions in a Farey Sequence are Farey neighbours they will remain so until their mediant separates them in a later Farey Sequence.

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Neighbour Property: Given $0 \leq \frac{a}{b} \leq \frac{c}{d} \leq 1$, $\frac{a}{b}$ and $\frac{c}{d}$ are Farey neighbours in Fn if and only if bc-ad = 1.

Proof: If $\frac{a}{b}$, $\frac{p}{q}$ and $\frac{c}{d}$ are in some Farey Sequence with $\frac{a}{b} < \frac{p}{q} < \frac{c}{d}$ and bp - aq = qc - pd = 1 then

bp + pd = qc + aq
p(b+d) = q(a+c)
$\frac{p}{q} = \frac{a+c}{b+d}$

Hence, $\frac{a}{b}$ and $\frac{p}{q}$ are neighbours and $\frac{p}{q}$ and $\frac{c}{d}$ are neighbours.

Conversely, we shall prove the result by induction.

For n=1, F1= $\{\frac{0}{1}, \frac{1}{1}\}$ and bc - ad= 1.

Hence result is true for n=1.

Let result is true for $(n-1)^{th}$ row. Any neighbour in n^{th} row will be $\frac{a}{b}$, $\frac{c}{d}$ or $\frac{a}{b}$, $\frac{a+c}{b+d}$ or $\frac{a+c}{b+d}$, $\frac{c}{d}$ where $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive fractions in $(n-1)^{th}$ row.
But then bc-ad = 1, b(c+a)-a(b+d)=bc-ad = 1, c(b+d)-(a+c)d = bc-ad =1.
Theorem is proved by induction.

Length of the Farey Sequence Fn: Length of the Farey Sequence Fn is given by the recursion formula:

$|Fn| = |\mathrm{Fn\text{-}1}| + \phi(n)$
where ϕ(n) is the Euler's Totient Function.

Application of Farey sequence for Rational Approximation of Irrational Numbers: Farey sequence can be utilized in the rational approximation of irrational numbers. Rational approximation of irrational numbers is representing irrational numbers with rational numbers. For example, there are many ways to represent √ 2. In its decimal notation we get 1.41421356237..., now we can turn that into a fraction, however the denominator gets quite large rather fast. The goal of rational approximation of irrational numbers is to represent an irrational number with a fraction with as small of a denominator as possible. For example, we can represent √ 2 with the fraction 7/5 . Although 7/ 5 is not equal to √ 2 it is within fourteen hundredths. Obviously the larger the denominator gets, the closer we will be able to come, however we will always strive for the most aesthetically pleasing answer. The Farey sequence can be used to prove theorems that state how close we can get to approximating rational numbers with an infinite amount of rational numbers.

Relationship of Farey Sequences With Ford Circles: Ford circles are a special case of mutually tangent circles; the base line can be thought of as a circle with infinite radius. Systems of mutually tangent circles were studied by Apollonius of Perga. A typical problem, which is presented on an 1824 tablet in the Gunma Perfective depicts the relationship of three touching circles with a common tangent. Given the size of the two outer large circles, then what is the size of small circle between them? The answer is equivalent to a Ford Circle :

$$\frac{1}{\sqrt{r_{middle}}} = \frac{1}{\sqrt{r_{left}}} + \frac{1}{\sqrt{r_{right}}}$$

Ford Circles are named after the American Mathematician Lenter R.Ford, Sr., who wrote about them in 1938.

There is a connection between Farey Sequence and Ford Circles. For every fraction $\frac{p}{q}$ (in its lowest terms), there is a Ford Circle C[p/q], which is the circle with radius $^1/_{2q^2}$ and centre at $(^p/_q, ^1/_{2q^2})$. These circles lie in half plane y≥0 and are tangent to the x- axis at the point $^p/_q$.

Two Ford circles for different fractions are either disjoint or they are tangent to one another- two Ford Circles never intersect. Interior of a Ford Circle contains no point of any other Ford Circle and the two Ford Circles C[a/b], C[a'/b'] are tangent if and only if $\frac{a}{b}$ and $\frac{a\prime}{b\prime}$ are adjacent Farey Fractions of same order.

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