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Rough Prime Ideals in Γ-Semirings

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Abstract

In this paper we introduce the notion of rough prime ideals, rough semi-prime ideals, strongly rough irreducible ideals and rough irreducible ideals of Γ-semirings and discuss some of its properties.

Key Words: Rough prime ideals, Rough semi-prime ideals, Rough maximal ideals, Strongly rough irreducible ideals and Rough irreducible ideals.

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1 Introduction

The notion of rough set was originally proposed by Pawlak[18,19] as a formal tool for modeling and processing incomplete data in information system. In rough set theory any subset of the universal set which has uncertainty or incomplete in nature is represented by two ordinary subsets. Some authors have studied the algebraic properties of rough set.

Several authors Chinram[3] and Thillagovindan et.al[22-24] studied rough sets in different algebraic structures. Davvaz[5] gave the relationship between rough set and ring theory. He considered a ring as the universal set and introduced the notion of rough ideals and rough subrings with respect to an ideal of a ring. Osman and Davvaz[17] studied the structure of rough prime(primary) ideals and rough fuzzy prime(primary) ideals in commutative rings.

The notion of Γ-semiring was introduced by Rao[20]. Characterization of ideals in semiring was were given by Ahsan[1], Iseki[12] and Shabir et. Al.[21]. Properties of prime and semi-prime ideals in Γ-semirings were discussed in detail by Dutta and Sardar [6-8]. Jagatap et.al.[13] discussed the right ideals, and maximal ideals of Γ-semirings. Hedayatiet.al.[11] introduced the notion of congruence relation in Γ-semiring.

2 Preliminaries

Definition 2.1

Let $\mathfrak{R}$ and Γ be two additive semigroups. $\mathfrak{R}$ is called Γ-semirings if there exists $\mathfrak{R}\times \Gamma \times \mathfrak{R} \rightarrow \mathfrak{R}$ denoted by $a\gamma b$, for all $a, b \in \mathfrak{R}$ and $\gamma \in \Gamma$ satisfying the following conditions:

(i) $a\gamma(b+c) = a\gamma b + a\gamma c$
(ii) $(b+c)\gamma a = b\gamma a + c\gamma a$
(iii) $a(\gamma_1+\gamma_2)c = a(\gamma_1 c)+(a\gamma_2 c)$
(iv) $a\gamma_1(b\gamma_2 c) = (a\gamma_1 b)\gamma_2 c$ for all $a, b, c \in \mathfrak{R}$ and $\gamma_1, \gamma_2 \in \Gamma$

Obviously semiring $\mathfrak{R}$ is a Γ-semiring.

Let $\mathfrak{R}$ be a semiring and Γ be a semigroup. Define a mapping $\mathfrak{R}\times \Gamma \times \mathfrak{R} \rightarrow \mathfrak{R}$ by $a\gamma b = ab$ for all $a, b \in \mathfrak{R}$ and $\gamma \in \Gamma$. Then $\mathfrak{R}$ is a Γ-semiring.

Definition 2.2. An element $0 \in \mathfrak{R}$ is said to be an *absorbing Zero* if $a\gamma 0 = 0 = 0\gamma a$ and $0 + a = a = a + 0$ for all $a \in \mathfrak{R}$ and $\gamma \in \Gamma$.

Definition 2.3: A nonempty subset T of $\mathfrak{R}$ is said to be *sub-Γ-semiring* of $\mathfrak{R}$ if $(T, +)$ is subsemigroup of $(\mathfrak{R}, +)$ and $a\gamma b \in T$ for all $a, b \in T$ and $\gamma \in \Gamma$.

Definition 2.4. A nonempty subset T of $\mathfrak{R}$ is called a *left*(resp. *right*) *ideal* of $\mathfrak{R}$ if $\mathfrak{R}$ is a subsemigroup of $(\mathfrak{R},+)$ and $x\gamma a \in T$ (resp. $a\gamma x \in T$) for all $a \in T, x \in \mathfrak{R}$ and $\gamma \in \Gamma$.
Definition 2.5. If T is both *left* and *right* i*deal* of $\mathfrak{R}$, then T is known as an ideal of $\mathfrak{R}$.

3 Congruence relation

In this section we introduce the notion of congruence relation and complete congruence relation in a Γ-semiring and study some of their properties. Throughout this paper $\mathfrak{R}$ denotes the Γ-semiring unless other wise mentioned.

Definition 3.1. Let θ be an equivalence relation on $\mathfrak{R}$. θ is called a *congruence relation* if $(a,b) \in \theta$ implies

(i) $(a+x, b+x) \in \theta$
(ii) $(x+a, x+b) \in \theta$
(iii) $(a\gamma x, b\gamma x) \in \theta$ and
(iv) $(x\gamma a, x\gamma b) \in \theta$, for all $x \in \mathfrak{R}$.

Even though the following theorem immediately follows from Definition 3.1, for sake of completeness a proof is provided.

Theorem 3.3.2. *Let θ be a congruence relation on a Γ-semiring $\mathfrak{R}$. Then $(a,b),(c,d) \in \theta$ implies $(a+c, b+d) \in \theta$, $(a\gamma c, b\gamma d) \in \theta$ for all $a,b,c,d \in \mathfrak{R}$.*

Proof. Let $(a,b),(c,d) \in \theta$. Since θ is a congruence relation $(a+c, b+c),(b+c, c+d) \in \theta$. This implies by transitive property in $\mathfrak{R}$, $(a+c, b+d) \in \theta$. Again $(a\gamma c, b\gamma c) \in \theta$ and $(b\gamma c, b\gamma d) \in \theta$ implies $(a\gamma c, b\gamma d) \in \theta$.

Lemma 3.3.3. *Let θ be a congruence relation on $\mathfrak{R}$. If $a,b \in \mathfrak{R}$, then*

(i) $[a]_\theta + [b]_\theta \subseteq [a+b]_\theta$
(ii) $[a]_\theta \Gamma [b]_\theta \subseteq [a\Gamma b]_\theta$

Proof. (i) Let $x \in R$. Suppose $x \in [a]_\theta + [b]_\theta$. Then there exist $y,z \in R$ such that $y \in [a]_\theta, z \in [b]_\theta$ and $x = y+z$. This means that $(a,y),(b,z) \in \theta$ and hence $(a+b, y+z) = (a+b, x) \in \theta$. Thus $x \in [a+b]_\theta$ and hence $[a]_\theta + [b]_\theta \subseteq [a+b]_\theta$.
(ii)Let $z = x\gamma y \in [a]_\theta \Gamma [b]_\theta$. Then $x \in [a]_\theta$ and $y \in [b]_\theta$. This implies that $(a,x) \in \theta$ and $(b,y) \in \theta$. Since θ is a congruence relation, $(a\gamma b, x\gamma y) \in \theta$.Thus $z = x\gamma y \in [a\Gamma b]_\theta$ and hence $[a]_\theta \Gamma [b]_\theta \subseteq [a\Gamma b]_\theta$.

A congruence relation θ on $\mathfrak{R}$ is called complete if $[a]_\theta + [b]_\theta = [a+b]_\theta$ and $[a]_\theta \Gamma [b]_\theta = [a\Gamma b]_\theta$

Definition 3.4. Let θ be a congruence relation on $\mathfrak{R}$ and A be a subset of $\mathfrak{R}$. Then the sets $\underline{\theta}(\mathrm{A}) = \{\mathrm{x} \in \mathfrak{R} \,/ [\mathrm{x}]_\theta \subseteq \mathrm{A}\}$ and $\overline{\theta}(\mathrm{A}) = \{\mathrm{x} \in \mathfrak{R} \,/ [\mathrm{x}]_\theta \cap \mathrm{A} \neq \phi\}$ are called the *lower* and *upper approximations* of the set A, respectively.

Let A be any subset of $\mathfrak{R}$. $\theta(A) = (\underline{\theta}(\mathrm{A}), \overline{\theta}(\mathrm{A}))$ is called a *rough set* with repect to θ if $\underline{\theta}(\mathrm{A}) \neq \overline{\theta}(\mathrm{A})$.

Lemma 3.5. *For any approximation space $(\mathfrak{R}, \theta)$ and $P \subseteq \mathfrak{R}$, the following hold:*

(1) $\underline{\theta}(\mathfrak{R} \setminus \mathrm{P}) = \mathfrak{R} \setminus \overline{\theta}(\mathrm{P})$
(2) $\overline{\theta}(\mathfrak{R} \setminus \mathrm{P}) = \mathfrak{R} \setminus \underline{\theta}(\mathrm{P})$
(3) $\overline{\theta}(\mathrm{P}) = \left(\underline{\theta}(\mathrm{P}^c)\right)^c$
(4) $\underline{\theta}(\mathrm{P}) = \left(\overline{\theta}(\mathrm{P}^c)\right)^c$

Theorem 3.6. *Let θ and ψ be congruence relations on $\mathfrak{R}$ and let A and B be nonempty subsets of $\mathfrak{R}$. Then*

(i) $\underline{\theta}(\mathrm{A}) \subseteq \mathrm{A} \subseteq \overline{\theta}(\mathrm{A})$

(ii) $\underline{\theta}(\emptyset) = \emptyset = \overline{\theta}(\emptyset)$

(iii) $\underline{\theta}(\mathrm{R}) = \mathrm{R} = \overline{\theta}(\mathrm{R})$

(iv) $\overline{\theta}(\mathrm{A} \cup \mathrm{B}) = \overline{\theta}(\mathrm{A}) \cup \overline{\theta}(\mathrm{B})$

(v) $\underline{\theta}(\mathrm{A} \cap \mathrm{B}) = \underline{\theta}(\mathrm{A}) \cap \underline{\theta}(\mathrm{B})$

(vi) $A \subseteq B$ implies $\underline{\theta}(\mathrm{A}) \subseteq \underline{\theta}(\mathrm{B})$ and $\overline{\theta}(\mathrm{A}) \subseteq \overline{\theta}(\mathrm{B})$

(vii) $\underline{\theta}(\mathrm{A} \cup \mathrm{B}) \supseteq \underline{\theta}(\mathrm{A}) \cap \underline{\theta}(\mathrm{B})$

(viii) $\overline{\theta}(\mathrm{A} \cap \mathrm{B}) \subseteq \overline{\theta}(\mathrm{A}) \cap \overline{\theta}(\mathrm{B})$

(ix) $\theta \subseteq \psi$ implies $\underline{\psi}(\mathrm{A}) \subseteq \underline{\theta}(\mathrm{A})$ and $\overline{\theta}(\mathrm{A}) \subseteq \overline{\psi}(\mathrm{A})$

(x) $(\overline{\theta \cap \psi})(\mathrm{A}) = \overline{\theta}(\mathrm{A}) \cap \overline{\psi}(\mathrm{A})$

(xi) $(\underline{\theta \cap \psi})(\mathrm{A}) \subseteq \underline{\theta}(\mathrm{A}) \cap \underline{\theta}(\psi)$

(xii) $\underline{\theta}\left(\underline{\theta}(\mathrm{A})\right) = \underline{\theta}(\mathrm{A})$

(xiii) $\overline{\theta}\left(\overline{\theta}(\mathrm{A})\right) = \overline{\theta}(\mathrm{A})$

(xiv) $\overline{\theta}\left(\underline{\theta}(\mathrm{A})\right) = \underline{\theta}(\mathrm{A})$

(xv) $\underline{\theta}\left(\overline{\theta}(\mathrm{A})\right) = \overline{\theta}(\mathrm{A})$

Definition 3.7. Let A be any subset of $\mathfrak{R}$ and $(\mathfrak{R}, \theta)$ be a rough approximation space. If $\underline{\theta}(\mathrm{A})$ and $\overline{\theta}(\mathrm{A})$ are ideals ,then $\underline{\theta}(\mathrm{A})$ is called a *lower rough ideal* and $\overline{\theta}(\mathrm{A})$ is called an *upper rough ideal* of $\mathfrak{R}$,respectively. $\theta(A) = (\underline{\theta}(\mathrm{A}), \overline{\theta}(\mathrm{A}))$ is called rough ideal of $\mathfrak{R}$.

Theorem 3.3.8. *Let θ be a congruence relation on $\mathfrak{R}$. If A is a left (resp.right) ideal of $\mathfrak{R}$, then $\overline{\theta}(A)$ is a left (resp.right) ideal of $\mathfrak{R}$.*

Proof. Let $a, b \in \overline{\theta}(\mathrm{A})$. Then $[a]_\theta \cap A \neq \emptyset, [b]_\theta \cap A \neq \emptyset$. So there exist $x \in [a]_\theta \cap A$ and $y \in [b]_\theta \cap A$. Since $x, y \in A, x + y \in A$. Now $x + y \in [a]_\theta + [b]_\theta \subseteq [a + b]_\theta$. Therefore $[a + b]_\theta \cap A \neq \emptyset$ and this means that $a + b \in \overline{\theta}(\mathrm{A})$.

Again let $x \in \overline{\theta}(\mathrm{A})$ and $r \in \mathfrak{R}$. Then there exists $y \in [a]_\theta \cap A$ and $(y, x) \in \theta$. Since θ is congruence relation , $(x\gamma r, y\gamma r), (r\gamma x, r\gamma y) \in \theta$. This means that $x\gamma r, r\gamma x \in \overline{\theta}(\mathrm{A})$. Thus $\overline{\theta}(\mathrm{A})$ is an ideal of $\mathfrak{R}$.

Theorem 3.3.9. *Let θ be a congruence relation on $\mathfrak{R}$. If A is a left (resp.right) ideal of $\mathfrak{R}$ and $\underline{\theta}(A)$ is nonempty, then $\underline{\theta}(A)$ is a left (resp.right) ideal of $\mathfrak{R}$.*

Proof. Let $a, b \in \underline{\theta}(\mathrm{A})$. Then $[a]_\theta, [b]_\theta \subseteq A$. Consider $[a + b]_\theta \subseteq [a]_\theta + [b]_\theta \subseteq A + A \subseteq A$. Thus $a + b \in \underline{\theta}(\mathrm{A})$.

Again let $a \in \underline{\theta}(\mathrm{A})$ and $r \in \mathfrak{R}$. Consider, $[a\gamma r]_\theta \subseteq [a]_\theta \Gamma [r]_\theta \subseteq A\Gamma R \subseteq A$ and $[r\gamma a]_\theta \subseteq [r]_\theta \Gamma [a]_\theta \subseteq R\Gamma A \subseteq A$. Thus $\underline{\theta}(\mathrm{A})$ is an ideal of $\mathfrak{R}$.

Corollary 3.3.10. *Let θ be a congruence relation on $\mathfrak{R}$. If A is an ideal of $\mathfrak{R}$and $\underline{\theta}(A)$ is nonempty, then $\theta(A) = (\underline{\theta}(A), \overline{\theta}(A)$ is a rough ideal of $\mathfrak{R}$.*

Lemma 3.3.11. *If I and J are ideals of $\mathfrak{R}$ and $\underline{\theta}(I \cap J)$ is a nonempty set, then $\left(\underline{\theta}(I \cap J), \overline{\theta}(I \cap J)\right)$ is a rough ideal of $\mathfrak{R}$.*

Theorem 3.3.12. *Let φ be an epimorphism of a semiring $\mathfrak{R}_1$ to a semiring $\mathfrak{R}_2$ and let θ_2 be a congruence relation on $\mathfrak{R}_2$. Then*

(i) $\theta_1 = \{(a,b) \in \mathfrak{R}_1 \times \mathfrak{R}_2 \,|\, (\varphi(a),\varphi(b)) \in \theta_2\}$ *is a congruence relation.*

(ii) *If θ_2 is complete and φ is 1-1, then θ_1 is complete.*

(iii) $\varphi(\overline{\theta_1}(A)) = \overline{\theta_2}(\varphi(A))$

(iv) $\varphi(\underline{\theta_1}(A)) \subseteq \underline{\theta_2}(\varphi(A))$

(v) *If φ is 1-1 , then* $\varphi\left(\underline{\theta_1}(A)\right) = \underline{\theta_2}(\varphi(A))$

Proof. (i) Let $(a,b) \in \theta_1$ and $x \in \mathfrak{R}_1$. Then $(\varphi(a),\varphi(b)) \in \theta_2$. Since θ_2 is a congruence relation ,we have $(\varphi(a)+\varphi(x),\varphi(b)+\varphi(x))$, $(\varphi(x)+\varphi(a),\varphi(x)+\varphi(b))$, $(\varphi(a).\varphi(x),\varphi(b).\varphi(x))$ and $(\varphi(x)\gamma\varphi(a),\varphi(x)\gamma\varphi(b))$ are in θ_2. φ being homomorphism , $(\varphi(a+x),\varphi(b+x))$, $(\varphi(x+a),\varphi(x+b))$, $(\varphi(a\gamma x),\varphi(b\gamma x))$ and $(\varphi(x\gamma a),\varphi(x\gamma b))$ are in θ_2. Again since φ being onto, $(a+x,b+x)$, $(x+a,x+b)$, $(a\gamma x,b\gamma x)$, $(x\gamma a,x\gamma b)$are in θ_1. Thus θ_1 is congruence relation in $\mathfrak{R}_1$.

(ii)Let θ_2 be complete . Assume that $z \in [a\gamma b]_{\theta_1}$. Then $(ab,z) \in \theta_1$. By definition of θ_2, $(\varphi(a\gamma b),\varphi(z)) \in \theta_2$. Hence

$$\begin{aligned}\varphi(z) &\in [\varphi(a\gamma b)]_{\theta_2}\\ &= [\varphi(a)\gamma\varphi(b)]_{\theta_2}\\ &= [\varphi(a)]_{\theta_2}\Gamma[\varphi(b)]_{\theta_2}\end{aligned}$$

Since $\varphi(z) \in [\varphi(a)]_{\theta_1}\Gamma[\varphi(b)]_{\theta_2}$, there exist $x,y \in \mathfrak{R}_1$ such that

$$\begin{aligned}\varphi(z) &= \varphi(x)\gamma\varphi(y)\\ &= \varphi(x\gamma y),\ \varphi(x) \in [\varphi(a)]_{\theta_2},\ \varphi(y) \in [\varphi(b)]_{\theta_2}\end{aligned}$$

Since φ is 1-1 and by definition of θ_1 , $z = x\gamma y$ and $x \in [a]_{\theta_1}$, $y \in [b]_{\theta_1}$. Thus $z \in [a]_{\theta_1}\Gamma[b]_{\theta_1}$ and therefore $[a\Gamma b]_{\theta_1} \subseteq [a]_{\theta_1}\Gamma[b]_{\theta_1}$. By Lemma 3.3, $[a]_{\theta_1}\Gamma[b]_{\theta_1} \subseteq [a\Gamma b]_{\theta_1}$. Hence $[a\Gamma b]_{\theta_1} = [a]_{\theta_1}\Gamma\,[b]_{\theta_1}$.

Again suppose $z \in [a+b]_{\theta_1}$.In the similar manner, one can get $[a+b]_{\theta_1} = [a]_{\theta_1} + [b]_{\theta_1}$. Thus θ_1 is complete.

(iii)Let $y \in \varphi(\overline{\theta_1}(A))$. Then there exists $x \in \overline{\theta_1}(A)$ such that $y = \varphi(x)$. This implies that $[x]_{\theta_1} \cap A \neq \emptyset$ and so there exists $a \in [x]_{\theta_1} \cap A$. Then $\varphi(a) \in \varphi(A)$ and $(a,x) \in \theta_1$ implies $(\varphi(a),\varphi(x)) \in \theta_2$. So $\varphi(a) \in [\varphi(x)]_{\theta_2}$. Thus $[\varphi(x)]_{\theta_2} \cap \varphi(A) \neq \emptyset$. This implies that $y = \varphi(x) \in \overline{\theta_2}(\varphi(A))$ and so

$$\varphi(\overline{\theta_1}(A)) \subseteq \overline{\theta_2}(\varphi(A)) \quad (1)$$

Again let $z \in \overline{\theta_2}(\varphi(A))$, there exists $x \in R_1$ such that $z = \varphi(x)$. Hence $[\varphi(x)]_{\theta_2} \cap \varphi(A) \neq \emptyset$. So there exists $a \in A$ such that $\varphi(a) \in \varphi(A)$ and $\varphi(a) \in [\varphi(x)]_{\theta_2}$. By definition of θ_1, we have $a \in [x]_{\theta_1}$. Thus $[x]_{\theta_1} \cap A \neq \emptyset$, which implies $x \in \overline{\theta_1}(A)$ and so $z = \varphi(x) \in \varphi(\overline{\theta_1}(A))$. It means that

$$\overline{\theta_2}(\varphi(A)) \subseteq \varphi(\overline{\theta_1}(A)) \quad (2)$$

From (1) and (2) the conclusion follows.

(iv)Let $y \in \varphi\left(\underline{\theta_1}(A)\right)$. Then there exists $x \in \underline{\theta_1}(A)$ such that $\varphi(x) = y$ and so we have $[x]_{\theta_1} \subseteq A$. Again let $b \in [y]_{\theta_2}$. Then there exists $a \in R_1$ such that $\varphi(a) = b$ and $\varphi(a) \in [\varphi(x)]_{\theta_2}$. Hence $a \in [x]_{\theta_1} \subseteq A$ and so $b = \varphi(a) \in \varphi(A)$. Thus $[y]_{\theta_2} \subseteq \varphi(A)$. This implies that

$y \in \underline{\theta_2}(\varphi(A))$ and so we have $\varphi\left(\underline{\theta_1}(A)\right) \subseteq \underline{\theta_2}(\varphi(A))$.

(v)Let $y \in \underline{\theta_2}(\varphi(A))$. Then there exists $x \in \mathfrak{R}_1$ such that $\varphi(x) = y$ and $[\varphi(x)]_{\theta_2} \subseteq \varphi(A)$. Let $a \in [x]_{\theta_1}$. Then $\varphi(a) \in [\varphi(x)]_{\theta_2}$ and so $a \in A$. Thus $[x]_{\theta_1} \subseteq A$ and $x \in \underline{\theta_1}(A)$. Hence $y \in \varphi(x) \in \varphi\left(\underline{\theta_1}(A)\right)$ and so we have $\underline{\theta_2}(\varphi(A)) \subseteq \varphi\left(\underline{\theta_1}(A)\right)$. By (iv), we have $\varphi\left(\underline{\theta_1}(A)\right) = \underline{\theta_2}(\varphi(A))$.

4. Rough Prime ideals in Γ-semirings

In this section we study the properties of prime ideals in a Γ-semiring using congruence and complete congruence relations. We have obtained characterizations of prime ideals in terms of the lower and upper approximations $\mathfrak{R}$.

Definition 4.1 An ideal P of $\mathfrak{R}$ is called a *prime ideal* of $\mathfrak{R}$ if $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$, for any ideals A and B of $\mathfrak{R}$

Definition 4.2: An ideal P of $\mathfrak{R}$ is called a *semi-prime* ideal if $A\Gamma A \subseteq P$ implies $A \subseteq P$ for any ideal A of $\mathfrak{R}$.

Obviously every prime ideal in $\mathfrak{R}$ is a semi-prime ideal.

Definition 4.3: An ideal I of $\mathfrak{R}$ is called an *irreducible ideal* if $A \cap B = I$ implies $A = I$ or $B = I$ for any ideals A and B of $\mathfrak{R}$.

Definition 4.4: An ideal I of $\mathfrak{R}$ is called a *strongly irreducible ideal* if $A \cap B = I$ implies $A \subseteq I$ or $B \subseteq I$ for any ideals A and B of $\mathfrak{R}$.

Definition 4.5: An ideal M of $\mathfrak{R}$ is said to be a *maximal ideal* of if there does not exist any other proper ideal of $\mathfrak{R}$ containing M properly.

Definition 4.6: Let θ be a congruence relation on $\mathfrak{R}$. An ideal P of $\mathfrak{R}$ is called a *rough prime ideal* of $\mathfrak{R}$ if $\overline{\theta}(P)$ and $\underline{\theta}(P)$ are prime ideals of $\mathfrak{R}$.

Definition 4.7: Let θ be a congruence relation on $\mathfrak{R}$. An ideal P of $\mathfrak{R}$ is called a *rough semi-prime ideal* of $\mathfrak{R}$ if $\overline{\theta}(P)$ and $\underline{\theta}(P)$ are semi-prime ideals of $\mathfrak{R}$.

Definition 4.8: An ideal I of $\mathfrak{R}$ is called an *rough irreducible* (*strongly irreducible*) *ideal* if if $\overline{\theta}(I)$ and $\underline{\theta}(I)$ are irreducible (strongly irreducible) ideal of $\mathfrak{R}$.

Theorem 4.9. *Let θ be a congruence relation on $\mathfrak{R}$ and P be a prime ideal of $\mathfrak{R}$ such that $\overline{\theta}(P) \neq \mathfrak{R}$, then $\overline{\theta}(P)$ is a prime ideal of $\mathfrak{R}$.*

Proof. Let P be prime ideal of $\mathfrak{R}$. By Theorem 3.3 $\overline{\theta}(P)$ is an ideal of $\mathfrak{R}$. Suppose A and B are ideals of $\mathfrak{R}$ such that $A\Gamma B \subseteq \overline{\theta}(P)$. Let $A \nsubseteq \overline{\theta}(P)$ and $\nsubseteq \overline{\theta}(P)$. Then there exists $a \in A$ such that $a \notin \overline{\theta}(P)$ and there exists $b \in B$ and $b \notin \overline{\theta}(P)$. This implies that $[a]_\theta \cap P = \emptyset$ and $[b]_\theta \cap P = \emptyset$ then $a, b \notin P$. This implies that $ab \notin P$ which is a contradiction to $A\Gamma B \subseteq P$. Thus $A \subseteq \overline{\theta}(P)$ or $B \subseteq \overline{\theta}(P)$ and so $\overline{\theta}(P)$ is a prime ideal of $\mathfrak{R}$.

Remark 4.10. As the conditions of the theorem are only necessary the concept of converse does not arise.

Theorem 4.11. *Let θ be a congruence relation on $\mathfrak{R}$ and P be a prime ideal of $\mathfrak{R}$. If $\underline{\theta}(P)$ is a non empty set, then $\underline{\theta}(P)$ is a prime ideal of $\mathfrak{R}$.*

Proof. Let P be prime ideal of $\mathfrak{R}$. By Theorem 3.3 $\underline{\theta}(P)$ is an ideal of $\mathfrak{R}$ such that $A\Gamma B \subseteq \underline{\theta}(P)$. Suppose that $A \nsubseteq \underline{\theta}(P)$ and $\nsubseteq \underline{\theta}(P)$. Then there exists $a \in A$ and $b \in B$ such that$[a]_\theta \nsubseteq P$ and

$[b]_\theta \nsubseteq P$. Then $a, b \notin P$. This implies that $ab \notin P$ which leads to a contradiction to the assumption $A\Gamma B \subseteq P$. This shows that $\underline{\theta}(P)$ is a prime ideal of $\mathfrak{R}$.

Remark 4.12. As the conditions of the theorem are only necessary the concept of converse does not arise.

Theorem 4.13. *An ideal P of $\mathfrak{R}$ is θ- upper rough prime ideal of $\mathfrak{R}$ if and only if $a\Gamma\mathfrak{R}\Gamma b \subseteq \overline{\theta}(P)$ implies $a \in \overline{\theta}(P)$ or $b \in \overline{\theta}(P)$ for any $a, b \in \mathfrak{R}$.*

Proof. Let P is a prime ideal of $\mathfrak{R}$. Then $\overline{\theta}(P)$ is a prime ideal of $\mathfrak{R}$. Let $a\Gamma\mathfrak{R}\Gamma b \subseteq \overline{\theta}(P)$ for any $a, b \in \mathfrak{R}$. Then $a\Gamma\mathfrak{R}\Gamma b\,\Gamma\mathfrak{R} \subseteq \overline{\theta}(P) \Rightarrow (a\Gamma\mathfrak{R})\,\Gamma(b\,\Gamma\mathfrak{R}) \subseteq \overline{\theta}(P)$ By $a\Gamma\mathfrak{R}$ and $b\,\Gamma\mathfrak{R}$ are ideals of $\mathfrak{R}$ and $\overline{\theta}(P)$ is a prime ideal of $\mathfrak{R}$. Therefore $a \in \overline{\theta}(P)$ or $b \in \overline{\theta}(P)$.

Conversely assume that the given statements holds . Let A and B be any two right ideal of $\mathfrak{R}$ such that ΓB $\subseteq \overline{\theta}(P)$. If $A \subseteq \overline{\theta}(P)$, then the result holds. Suppose that $A \nsubseteq \overline{\theta}(P)$. Hence there exists an element $a \in A$ such that $a \notin \overline{\theta}(P)$. For any $b \in B$, $a\Gamma\mathfrak{R}\Gamma b = (a\Gamma\mathfrak{R})\Gamma\, b \subseteq A\Gamma B \subseteq \overline{\theta}(P)$. Therefore by assumption $b \in \overline{\theta}(P)$ implies $B \subseteq \overline{\theta}(P)$.

Therefore P is a θ- upper rough prime ideal of $\mathfrak{R}$.

Theorem 4.14. *An ideal P of $\mathfrak{R}$ is θ- lower rough prime ideal of $\mathfrak{R}$ if and only if $a\Gamma\mathfrak{R}\Gamma b \subseteq \underline{\theta}(P)$ implies $a \in \underline{\theta}(P)$or $b \in \underline{\theta}(P)$ for any $a, b \in \mathfrak{R}$.*

Proof. The proof is similar to the above theorem. Hence omitted.

Theorem 4.15. *An ideal P of $\mathfrak{R}$ is θ- upper rough semi-prime ideal of $\mathfrak{R}$ if and only if $a\Gamma\mathfrak{R}\Gamma a \subseteq \overline{\theta}(P)$ implies $a \in \overline{\theta}(P)$ for any $a \in \mathfrak{R}$.*

Proof. Suppose that P is a prime ideal of $\mathfrak{R}$. Then $\overline{\theta}(P)$ is a semi-prime ideal of $\mathfrak{R}$. Let $a\Gamma\mathfrak{R}\Gamma\, a \subseteq \overline{\theta}(P)$ for $a \in \mathfrak{R}$. Then $a\Gamma\mathfrak{R}\Gamma\, a\Gamma\mathfrak{R} \subseteq \overline{\theta}(P) \Rightarrow (a\Gamma\mathfrak{R})\Gamma(\,a\Gamma\mathfrak{R}) \subseteq \overline{\theta}(P)$. By $a\Gamma\mathfrak{R}$ is a ideal of $\mathfrak{R}$ and $\overline{\theta}(P)$ is a semi-prime ideal of $a\Gamma\mathfrak{R} \subseteq \overline{\theta}(P)$.

Conversely assume that given statements holds. Let A be any ideal of $\mathfrak{R}$ such that ΓA $\subseteq \overline{\theta}(P)$. For any $a \in A$, $(a\Gamma\mathfrak{R})\Gamma\, a \subseteq A\Gamma A \subseteq \overline{\theta}(P)$. Therefore by assumption $a \in \overline{\theta}(P)$ implies $A \subseteq \overline{\theta}(P)$. Hence $\overline{\theta}(P)$ is a semi-prime ideal of $\mathfrak{R}$.

Theorem 4.16. *An ideal P of $\mathfrak{R}$ is θ- lower rough semi-prime ideal of $\mathfrak{R}$ if and only if $a\Gamma\mathfrak{R}\Gamma a \subseteq \underline{\theta}(P)$ implies $a \in \underline{\theta}(P)$ for any $a \in \mathfrak{R}$.*

Proof. The proof is similar to the above theorem. Hence omitted.

Theorem 4.17. *Every semi-prime and strongly irreducible ideal of $\mathfrak{R}$ is a rough prime ideal of $\mathfrak{R}$.*

Proof. Let P be semi-prime and strongly irreducible ideal of $\mathfrak{R}$. Then $\overline{\theta}$(P) is a semi-prime and strongly irreducible ideal of $\mathfrak{R}$. Let A and B be any two ideals of $\mathfrak{R}$, then $A\Gamma B \subseteq \overline{\theta}(P)$. $(A \cap B)2 \subseteq (A\cap B)\Gamma(A\cap B) \subseteq A\Gamma B \subseteq \theta P$. By P is a semi-prime ideal, $(A\cap B) \subseteq \theta P$. Therefore $A \subseteq \theta P$ or $B \subseteq \overline{\theta}(P)$, since $\overline{\theta}$(P) is a semi-prime and strongly irreducible ideal of $\mathfrak{R}$. Thus $\overline{\theta}$(P) is a prime ideal of $\mathfrak{R}$.

Similarly $\underline{\theta}(P)$ is also a prime ideal of $\mathfrak{R}$. Hence $\theta(P)$ is a rough prime ideal of $\mathfrak{R}$.

Theorem 4.18. *Any maximal ideal of $\mathfrak{R}$ is a rough prime ideal of $\mathfrak{R}$.*

Proof. Let M be any maximal ideal of $\mathfrak{R}$. Then $\overline{\theta}$(M) is a maximal ideal of $\mathfrak{R}$. Suppose that $a \notin \overline{\theta}$(M). $a\Gamma\mathfrak{R}$ is a ideal of $\mathfrak{R}$ which contains an element a. By $\overline{\theta}$(M) is a maximal ideal, $\overline{\theta}(\mathrm{M}) + a\Gamma\mathfrak{R} = \mathfrak{R}$. As $1 \in \mathfrak{R}$, $1 \in m + \sum_i a\gamma_i x_i$. Then $1\gamma b = m\gamma b + \left(\sum_i a\gamma_i x_i\right)\gamma b \subseteq \overline{\theta}(\mathrm{M}) + a\Gamma\mathfrak{R}\Gamma \mathrm{b} \subseteq \overline{\theta}(\mathrm{M})$. Therefore $\in \overline{\theta}$(M) . This shows that $\overline{\theta}$(M) prime ideal of $\mathfrak{R}$.

Similarly $\underline{\theta}(M)$ is prime ideal of $\mathfrak{R}$. Hence $\theta(M)$ is a rough prime ideal of $\mathfrak{R}$.

5 Conclusion.

The purpose of this paper is to build a connection between rough set theory and Γ-semiring. We have introduced the notions of lower and upper approximation subsets of a Γ-semiring and characterized the prime ideals of a Γ-semiring through lower and upper approximations. This definition and results can be extended to other algebraic structures such as rings and modules.

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