# Probability and its Calculation Terms

**Priya, Department of Mathematics, KUK**

**Abstract:** In the topic study group on probability at ICME 11 a variety of ideas on probability education were presented. Some of the papers have been developed further by the driving ideas of interactivity and use of the potential of electronic publishing. As often happens, the medium of research influences the results and thus -not surprisingly -the research change its character during this process. This paper provides a summary of the main threads of research in probability education across the world and the result of an experiment in electronic communication. For convenience of international readers, abstracts in Spanish and German have been supplied, as well as hints for navigation to linked electronic materials.

ISSN : 2348-5612 © URR
9 770234 856124

**Background:** Whether probability education needs to be seen as discrete and separate from statistics has been an ongoing debate for many decades. Nowadays, statistics seems to be dominant in school education and data handling has been a key theme as part of the movement of mathematics for all. Conversely probability is thought to be harder and less relevant. Nevertheless, probability is an important discipline in its own right, and does contain the key underpinning concepts to understand and use data sensibly. This paper focuses on the international research on probability issues in education, mainly derived from ICME 11 (2008) in Mexico. This study group, also linked to IASE included all the major themes being studied internationally in probability education. A thorough system of peer review, using an international panel of experts, ensured that the papers at ICME 11 were both well written and also covered the key areas of research being undertaken across the world. The full papers from the conference are available on the ICME 7 site in the links below. Most of the papers have been developed further, including many interactive features (such as links to related and background research ideas) and can be found in Borovcnik & Kapadia (2009)[1], which can also be accessed electronically. These themes of international research provide a valuable bridge between international research and themes in Britain on the teaching of probability, which is often subsumed under data handling or statistics. Probability education has not been a central focus of the research community in the last three decades since the theoretical framework espoused by Kapadia & Borovcnik (1991)[2] appeared. Jones (2005) on "Exploring probability in Schools" has largely followed the new paradigm in educational research, which is empirically oriented. Designs of teaching sequels are administered to students and analysed. Sometimes, beliefs and attitudes of teachers are empirically investigated. Only one of the contributions in Jones is philosophically oriented: Batanero, Henry, & Parzysz (2005)[3] give a summary of the philosophical debate on the interpretations of probability and discuss its implications on teaching. Analysis of the subject matter is still dominated by Heitele's fundamental ideas (1975)[4], which seem to be more a description of the main chapters of a probability textbook than an analysis of the concepts from a more general perspective and their purpose. The educational debate is being revived by the more recent endeavours to explain the concepts of risk (see Gigerenzer 2002)[5], which comes from societal needs taken up by cognitive psychologists and thereby attracts more attention in the community of educationalists. However, the fundamental ideas of probability, in line with those discussed for statistics by Wild & Pfannkuch (1999) are still awaiting elaboration. Some starting points may be found in Pratt (2005)[6] though the CERME working group, too, is mainly devoted to the empirical paradigm. The authors support the ideas and results of Fischbein (1987)[7] who elaborates on intuitions and their impact on understanding (and accepting) probabilistic concepts. Raw primary intuitions of individuals are revised

[1] Borovcnik & Kapadia (2009), ). Judgement under Uncertainty: Heuristics and Biases. Cambridge University Press.

[2] Kapadia & Borovcnik (1991)[2], 'Bayesian Statistics: An Introduction,

[3] Quite a good account of Bayesian statistics, but still not for the feint hearted (Lindley's book is better for beginners).

[4] Heitele's fundamental ideas (1975)[4], . Something of a classic book. Argues that Bayesian probability is the natural basis for decision making in all walks of life.

[5] Wilson A.G. Cognitive Factors Affecting Subjective Probability Assessment. Institute of Statistics and Decision Sciences Discussion Paper No. 94-02. Duke University, Durham, USA (1994).

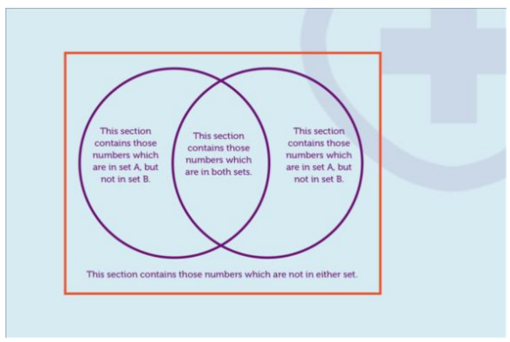
[6] Pratt (2005)[6] . Cognitive Factors Affecting Subjective Probability Assessment. Institute of Statistics and Decision Sciences Discussion Paper No. 94-02. Duke University, Durham,

[7] Fischbein (1987)[7] Subjective Probability. John Wiley & Sons Ltd., (1994).

by teaching interventions and changed to secondary intuitions, which should and could help to handle with the formal sides of the concepts as a companion. Such ideas have the potential to describe any learning process; research begins with the learning process of an individual or a group on the research topic of interest and later becomes a learning process within the wider community when research papers are published, Fischbein's ideas might also orientate the way a research community exchanges its results and enriches the discussion and makes progress. Despite the fact that a multitude of new technologies is available now in the era of information technology, and multi-media is spreading to all corners of life, publication in research has hardly changed. Textbooks for study are changing gradually, a few hypertexts make use of the possibilities of new media but for research publications it seems that times have not changed yet. Borovcnik (2007) has analysed the consequences of new technologies on applications and on educational endeavours; more endeavour is needed from the research community to improve its communication. Kapadia & Borovcnik (2009)[8] have taken up the challenge of innovative publication (including multi-media and more) in the age of web 2.0. In what follows, we will focus on the content, the main streams of research in educational probability by classifying the international endeavours to obtain new insights on the teaching of probability.

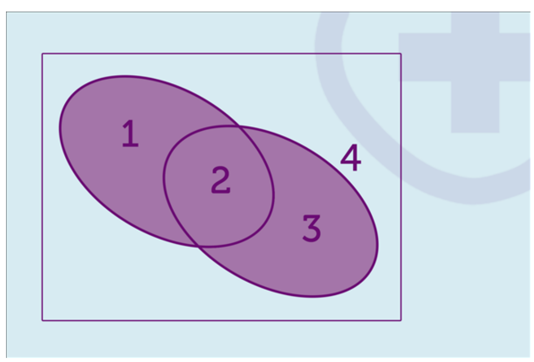
**Using Venn diagrams to calculate probabilities:** A **'Venn diagram'** is another useful way to represent mathematical or logical sets of information. In a Venn diagram, the position and overlapping of circles are used to indicate the relationships between different sets of information.[9]

For example, a Venn diagram can be used to display how many people own a cat 'A', how many people own a dog 'B', how many people own both a cat and a dog 'A and B' and how many people own neither a cat nor a dog 'not A and not B':




There are several terms used to describe the different sets within Venn diagrams:

**A U B** or **"A union B"** refers to everything which is included in either of the sets. For example, in the following diagram, the answers are 1, 2, and 3.



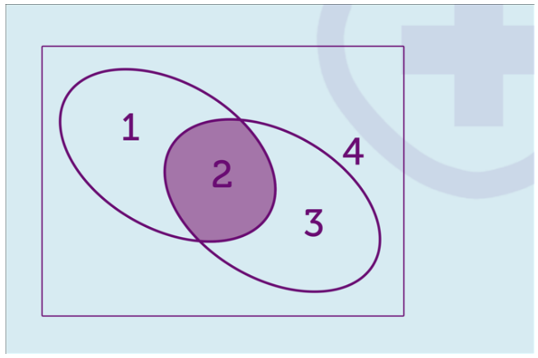

**A ∩ B** or **"A intersect B"** refers only to the values which are in both of the sets. For example, in the following diagram, the answer is 2.

---

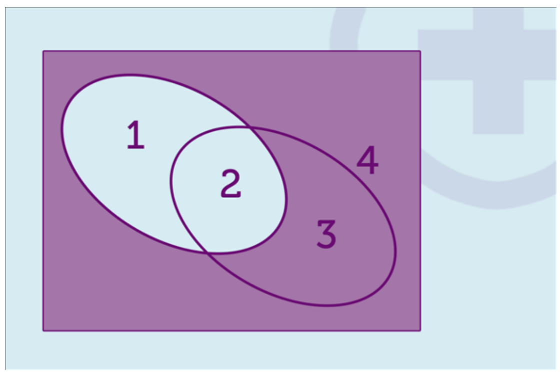
[8] Kapadia & Borovcnik (2009)[8] "An Introduction to Probability Theory and Its Applications", (Vol 1), , ISBN 0-471-25708-7.

[9] William Feller, "An Introduction to Probability Theory and Its Applications", (Vol 1), 3rd Ed, (1968), Wiley, ISBN 0-471-25708-7.

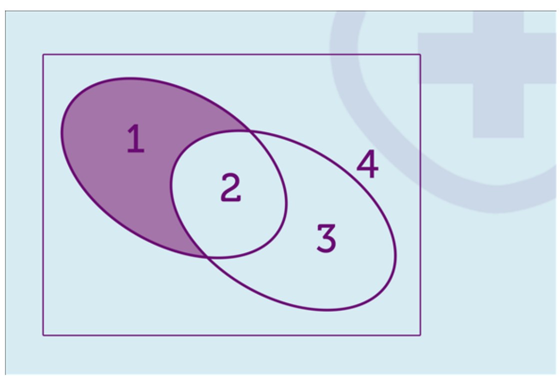
ISSN : 2348 - 5612 | Volume : 04 , Issue : 13 | October – December 2017




**Ac or ~A** or **"not A"** refers to everything outside of the region of A. For example, in the diagram below, the answers are 3 and 4.




**A – B** or **"A minus B"** refers to everything in the region of A, with the exception of anything which is in its overlap with B. For example, in the following diagram the answer is 1.




**Central themes in international probability education research:** Probability and statistics have been part of school mathematics for less than 40 years and complement the traditional topics of arithmetic, algebra and geometry. Statistics is part of the curriculum in virtually all countries but ideas of probability may only be introduced for older pupils. Application-oriented statistics is undisputed in its relevance, but the place of probability is more ambivalent. Reduction of probability to the classical conception, mainly based on combinatory, and its perception as a solely mathematical discipline with its connection to higher mathematics, are sometimes used as arguments to abandon it in favour of the statistics part. However, there are key reasons for a strong role for probability within mathematics curricula:[10]

[10] *Finetti, Bruno de (1970). "Logical foundations and measurement of subjective probability". Acta Psychologica.* ***34****: 129–145.*

1. Misconceptions on probability affect people's decisions in important situations, such as medical tests, jury verdicts, investment, assessment, etc.
2. Probability is essential to understand any inferential procedure of statistics.
3. Probability offers a tool for modelling and "creating" reality, such as in physics.
4. The concepts of risk (not only in financial markets) and reliability are closely related to and dependent upon probability.
5. Probability is an interesting subject in its own right and worthy of study.

The challenge is to teach probability in order to enable students to understand and apply it, by creating approaches that are both accessible and motivating. Both, the frequentist and subjectivist views of probability, and connections of probability to practical applications should be taken into account. Simulation is one such strategy, as is visualisation of abstract concepts. The use of technology helps to reduce the technical calculations and focus the learner on the concepts instead. The world of personal attitudes and intuitions is another source for success or failure of teaching probability. The main themes in the research, which nevertheless do overlap are: Conditional Probability and Bayes' Theorem; The School Perspective: Pre- and Misconceptions; The Teachers' Perspective: Pre- and in-service Courses; Impact of Technology; Fundamental Ideas.

**Conditional Probability and Bayes' Theorem:** Conditional probability and Bayesian inference are important ingredients of university teaching, including courses for non-mathematical students. Many different types of errors have been investigated in isolation. According to C. Batanero and C. Diaz (Spain), there is neither a study investigating connections between various types of misconceptions, nor an analysis whether misconceptions are related to mathematical knowledge, i.e. whether they decrease with better achievement in mathematics. Consequently, they have developed a test with (mainly familiar) items, and administered it to university students. Results were analysed with factor analysis. Though some phenomena remained even with higher mathematics education, there was a significant decrease in misconceptions with a higher level of mathematics. For interrelations between several misconceptions, the results were less optimistic as these misconceptions seemed to be quite isolated. As a consequence, Endeavour in probability education has to be fostered and misconceptions still have to be continually and repeatedly stressed in teaching in order to facilitate students' understanding.[11]

P. Huerta (Spain) identifies a serious flaw of some existing research which does not take the structure of the posed problems into account. He has classified the mathematical structure of "ternary problems" into 20 different types of problems with conditional probabilities of which only one subclass (and from it mainly one type of task) has been used in existing research. A graph with all problems has been developed to visualise the grade of difficulty of a special problem at hand. Applying this deep structural analysis, Huerta has developed a plan for future empirical research to cover all types of conditional probability problems to enhance the insights which might be gained.[12]

**The School Perspective: Pre- and Misconceptions:** There has been a trend away from misconceptions, which may be changed by suitable teaching, towards pre-conceptions. Such a change of focus in research may be traced in current empirical research.

D. Abrahamson (USA) designed an experiment with a single child (Li, 11 years), using in an in-depth interview after a teaching phase in a classroom environment where an urn experiment was replaced step by step by the computer environment. Abrahamson analysed the learning trajectory of the child and how the interaction of the representation of the notions by different media influenced learning (see below), starting with a 4-block scooper to pick out blue and green marbles, as illustrated below. As a valuable side effect of the approach, the histogram can be visually linked (with a "greenish" impression) to the proportion of green marbles in the urn.

**The Teachers' Perspective: Pre- and in-service Courses:** There are pitfalls in the interpretation of results from statistical tests or from confidence intervals. These originate from the reduction of the interpretation of probability

---

[11] *Moore, W.J. (1992). Schrödinger: Life and Thought. Cambridge University Press. p. 479. ISBN 978-0-521-43767-7.*

[12] Franklin, J. (2001) *The Science of Conjecture: Evidence and Probability Before Pascal*,Johns Hopkins University Press. (pp. 22, 113, 127).

to situations, which may be repeated independently in the same manner. On this issue there has been a vigorous debate not only in the foundations of statistics but also in the didactical community. Ö. Vancsó (Hungary) has developed a parallel course in classical and Bayesian statistics. He believes that it is a false dichotomy, to teach either classical statistics or Bayesian statistics, as both offer a consistent theory of probability. He has tried and refined his ideas in several cycles in teacher pre- service education and reported his positive experiences: "Now I have really understood what is meant by confidence intervals" one of his students exclaims. An interesting extra-curricular activity was explored by H. Trevethan (Mexico) who described a project in the context of a science fair. A pair of students worked on a project to present a game of chance. There were several advantages. These included the autonomous activity of the students, their own responsibility, presenting in public etc. The game was "Shut the box", which is certainly open to varied stochastic strategies as different dice can be chosen to play the game. This authentic (and not artificial) transfer of responsibility could well be taken up more often in teaching in class. Mathematically, conditional probabilities and Bayes' theorem are the key concepts to develop winning strategies.[13]

K. Lyso (Denmark)[14] used a battery of standard tasks covering the main primitive concepts including two-stage experiments. The distinctive feature was on the discussion about which solutions are feasible, or which reconstruction of the task made sense and therefore led to a sensible solution even if it did not coincide with the "normative" solution. One result was the documentation of an inclination to reformulate tasks. The students sometimes reformulated two-stage experiments into one-stage tasks getting a wrong answer but found it hard to see why their reconstruction was misleading:

L. Zapata (Columbia/USA)[15] investigated well-known tasks from Kahneman & Tversky in order to clarify what may be learned from new as well as more experienced teachers. She has tried to derive meta-knowledge from her in-depth interviews with teachers. Surprisingly, or possibly unsurprisingly, novice teachers repeated the same misleading intuitive conceptions as their students and thus were not really able to help them. It may be that probability is much more prone to such difficulties than other topics in mathematics. This is confirmed by V. Kataoka (Brazil)[16] who ran a series of workshops of in-service education. One special experiment used in the workshop illustrates the importance of suitable models and data sampled by randomness (when do you really have data from random samples?)

Success rates of 75% are not rare. In contrast to it, there are (at least) two models for randomly breaking the stick (with 25% and 19% success rates). The obvious discrepancy between the theory and the model lets us gradually start to doubt whether we can break the stick truly randomly into 3 parts. As a conclusion, relative frequencies might sometimes be of no value to estimate an unknown probability. This enriches the usual discussion about the convergence of relative frequencies by focussing on the underlying assumption of randomness of the data. Analogous examples are abundant but are less emotionally laden than spaghetti.

**Impact of Technology:** Technology can be viewed in at least two very distinct ways. In one aspect is the media used such as Powerpoint or interactive use of computers by students. The other aspect relates to the software tools used. Some software is generic (eg Excel) and some software is designed specifically for probability such as Fathom. In practice there is more software relating to statistics, though probability software is growing. Fathom and Tinkerplot can be used for efficient calculations and for illustrating key ideas such as the concept of distribution and the law of large numbers, as noted by S. Inzunsa (Mexico) and R. Peard (Australia).[17]

New media indirectly form the backbone of the research of D. Pratt, writing with R. Kapadia (England) on shaping the experience of naïve probabilists. Sequences of the programme ChanceMaker supplied new and challenging

---

13 *"Data: Data Analysis, Probability and Statistics, and Graphing". archon.educ.kent.edu. Retrieved 2017-05-28.*

14 K. Lyso (Denmark)[14] "Whither Efficient Markets? Efficient Market Theory and Behavioral Finance". The Finance Professionals' Post, 2010.

15 L. Zapata (Columbia/USA)[15] *"Andrei Nikolaevich Kolmogorov". CWI Quarterly (1): 3–18. Retrieved 27 January 2016.*

16 http://www.statslab.cam.ac.uk/~rrw1/markov/M.pdf

17 Franklin, James (2001). *The Science of Conjecture: Evidence and Probability Before Pascal*. Johns Hopkins University Press. ISBN 978-0801865695.

experiences to learners in order to shape their intuitions and strategies. There are new challenges for designers of software and teachers using this software. In a fusion of control over the initial parameters (via randomness) and representations of results (histograms for the distribution of data or statistics like the mean), new insights into randomness have been generated. A new world of up-to-date unknown intuitions might emerge, which would affect concepts and their understanding.

J. Watson and S. Ireland (Australia) reported the results of in-depth interviews covering issues on the relations between empirical and theoretical aspects of probability. The class of 12 year olds undertook coin tossing and tabulation of results, followed similar experiments with Tinkerplot, which is becoming a popular piece of software. This widened the children's experience. Some questions remained open for further scrutiny. Can the computer really generate randomness? How can one read diagrams from the software correctly? How can we ensure that the children have sufficient experience in proportional thinking?

**Fundamental Ideas:** The fundamental ideas of probability include random variables, distribution, \ expectation, and relative frequencies, as well as the central limit theorem. R. Peard (Australia) has gone to the roots of the subject with questions from games of chance. Games of chance have been partially discredited by their closeness to combinatorics (which is not always easy to understand) and by their artificiality (we want to teach real applications to our students). However, games of chance have spread widely, such as lotteries, and developed to become an important business sector, which is still growing.

**Perspectives for the future:** We end with some assertions requiring further research Endeavour in probability education:

- People use their experience in order to judge probabilities incompletely and – even worse – in a haphazard manner.
- People have difficulties in judging very small and very high probabilities especially if these are connected to adverse consequences.
- People are inclined to attribute equal chances to the – given or seen – possibilities, especially if there are just two.
- People attribute probabilities and process these into new ones neglecting even the most basic rules (e. g. all probabilities sum to 1).

We believe that sharing and testing ideas across different countries will help promote deeper understanding. In particular, further empirical testing using shared instruments will yield deeper insights.

**Reference:**


- [1] Borovcnik & Kapadia (2009), ). Judgement under Uncertainty: Heuristics and Biases. Cambridge University Press.
- [1] Kapadia & Borovcnik (1991)[1], 'Bayesian Statistics: An Introduction,
- [1] Quite a good account of Bayesian statistics, but still not for the feint hearted (Lindley's book is better for beginners).
- [1] Heitele's fundamental ideas (1975)[1], . Something of a classic book. Argues that Bayesian probability is the natural basis for decision making in all walks of life.
- [1] Wilson A.G. Cognitive Factors Affecting Subjective Probability Assessment. Institute of Statistics and Decision Sciences Discussion Paper No. 94-02. Duke University, Durham, USA (1994).
- [1] Pratt (2005)[1] . Cognitive Factors Affecting Subjective Probability Assessment. Institute of Statistics and Decision Sciences Discussion Paper No. 94-02. Duke University, Durham,
- Wright G. & Ayton P. (eds). Subjective Probability. John Wiley & Sons Ltd., (1994)
- "Kendall's Advanced Theory of Statistics, Volume 1: Distribution Theory", Alan Stuart and Keith Ord, 6th Ed, (2009), ISBN 9780534243128.
- William Feller, "An Introduction to Probability Theory and Its Applications", (Vol 1), 3rd Ed, (1968), Wiley, ISBN 0-471-25708-7.
- Probability Theory The Britannica website.

- *Finetti, Bruno de (1970). "Logical foundations and measurement of subjective probability". Acta Psychologica. **34**: 129–145.* doi:10.1016/0001-6918(70)90012-0.
- *Jaynes, E. T. (2003-06-09). "Section 5.3 Converging and diverging views". In Bretthorst, G. Larry. Probability Theory: The Logic of Science (1 ed.). Cambridge University Press.* ISBN 9780521592710.
- Franklin, J. (2001) *The Science of Conjecture: Evidence and Probability Before Pascal,*Johns Hopkins University Press. (pp. 22, 113, 127).
- *Moore, W.J. (1992). Schrödinger: Life and Thought. Cambridge University Press. p. 479. ISBN 978-0-521-43767-7.*
- Franklin, James (2001). *The Science of Conjecture: Evidence and Probability Before Pascal*. Johns Hopkins University Press. ISBN 978-0801865695.
- [1] *"Data: Data Analysis, Probability and Statistics, and Graphing". archon.educ.kent.edu. Retrieved 2017-05-28.*
- [1] Singh, Laurie (2010) "Whither Efficient Markets? Efficient Market Theory and Behavioral Finance". The Finance Professionals' Post, 2010.
- [1] *Vitanyi, Paul M.B. (1988). "Andrei Nikolaevich Kolmogorov". CWI Quarterly (1): 3–18. Retrieved 27 January 2016.*
- [1] http://www.statslab.cam.ac.uk/~rrw1/markov/M.pdf